A NONLINEAR OBSERVER FOR GYRO ALIGNMENT ESTIMATION

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A nonlinear observer for gyro alignment estimation is presented. The observer is composed of two error terms, an attitude error and an alignment error. The observer is globally stable with exponential convergence of the attitude errors. The gyro alignment estimate converges to the true alignment when the system is completely observable.

INTRODUCTION

Combined observer-controller designs for the attitude control of rigid flight vehicles are a subject of active research[1]. Successful design of such architectures is complicated by the fact that there is, in general, no separation principle for nonlinear systems. In contrast to linear systems, "certainty equivalence" substitution of the states from an exponentially converging observer into a nominally stabilizing, state feedback control law does not necessarily guarantee stable closed-loop operation for the coupled systems[2,3].

One version of this problem, in particular, is the task of forcing the attitude of a rigid vehicle to asymptotically track a (time-varying) reference attitude using feedback from rate sensors with alignment errors. Alignment errors are typically estimated using a least-squares approach, or an extended Kalman filter[4-8]. Here, in order to determine the misalignment errors, we propose utilizing an angular velocity observer similar to the observers in refs.1 and 9, using the estimated misalignment state in the nonlinear observer presented in ref. 10. In the analysis below we demonstrate that the resulting system provides stable estimation of the alignment states.

The main proof proceeds in two steps. First, we extend the analysis of ref. 1 to the case of gyro misalignment first for a spacecraft with a constant angular velocity. Given the necessary observability conditions, we demonstrate that the alignment estimate converges to the true alignment. We extend the analysis for the case of a time varying angular velocity. The proof in ref. 1 uses a Lyapunov argument to obtain a similar result, but under the assumption that the alignment errors are small; this restriction is removed here.

DEFINITIONS

The attitude of a spacecraft can be represented by a quaternion, consisting of a unit rotation vector **e**, known as the Euler axis, and a rotation ϕ about this axis, given as

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$$\mathbf{q} = \begin{bmatrix} e\sin\frac{\phi}{2} \\ \cos\frac{\phi}{2} \end{bmatrix} = \begin{bmatrix} \epsilon \\ \eta \end{bmatrix}$$

where $\mathbf{q}$ is the quaternion, partitioned into a vector part, ϵ and a scalar part, η. Typically, in spacecraft attitude applications, the quaternion represents the rotation from an inertial coordinate system to the spacecraft body coordinate system. Note that $\|\mathbf{q}\| = 1$ by definition. The rotation matrix for a specific attitude can be computed as[11]

$$R(\mathbf{q}) = (\eta^2 - \epsilon^T\epsilon)I + 2\epsilon\,\epsilon^T - 2\eta S(\epsilon) \tag{1}$$

where $S(\epsilon)$ is a matrix representation of the vector cross product operation.

$$S(\epsilon) = \begin{bmatrix} 0 & -\varepsilon_z & \varepsilon_y \\ \varepsilon_z & 0 & -\varepsilon_x \\ -\varepsilon_y & \varepsilon_x & 0 \end{bmatrix}$$

A rotation between coordinate frames is computed as[12]

$$\tilde{\mathbf{q}} = \begin{bmatrix} \tilde{\epsilon} \\ \tilde{\eta} \end{bmatrix} = \mathbf{q}_1 \otimes \mathbf{q}_2^{-1} = \begin{bmatrix} \eta_2 I - S(\epsilon_2) & -\epsilon_2 \\ \epsilon_2^T & \eta_2 \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \eta_1 \end{bmatrix} \tag{2}$$

Where $\tilde{\mathbf{q}}$ defines the rotation from the frame defined by $\mathbf{q}_2$ to the frame defined by $\mathbf{q}_1$. Note that $\|\tilde{\epsilon}\| = 0,\ \tilde{\eta} = \pm 1$ indicates that frame 2 is aligned with frame 1.

The kinematic equation for the quaternion is given as

$$\dot{\mathbf{q}}(t) = \begin{bmatrix} \dot{\epsilon} \\ \dot{\eta} \end{bmatrix} = \tfrac{1}{2} Q(\mathbf{q}(t)) R_g(\mathbf{q}_g)\omega_g(t) \tag{3}$$

where

$$Q(\mathbf{q}(t)) = \begin{bmatrix} \eta(t)I + S(\epsilon(t)) \\ -\epsilon(t)^T \end{bmatrix} = \begin{bmatrix} Q_1(\mathbf{q}(t)) \\ -\epsilon(t)^T \end{bmatrix} \tag{4}$$

The kinematic equation for an attitude matrix is[11]

$$\dot{R}(\mathbf{q}(t)) = -S(R_g(\mathbf{q}_g)\omega_g(t))R(\mathbf{q}(t)) \tag{5}$$

$\omega_g(t)$ is the spacecraft angular velocity with respect to inertial space, in gyro coordinates, transformed to body coordinates with the alignment matrix $R_g(\mathbf{q}_g)$. $R_g(\mathbf{q}_g)$ is computed from (1) above with the constant gyro alignment quaternion given as

$$\mathbf{q}_g = \begin{bmatrix} \epsilon_g \\ \eta_g \end{bmatrix}$$

The angular velocity is typically measured by a gyro, which can be corrupted by various errors, such as bias, alignment, scale factor errors, and noise. In this work we consider the case of alignment errors, where the matrix $R_g(\mathbf{q}_g)$ is unknown.

NONLINEAR OBSERVER

Following the development in Ref. 10, observers for the attitude and gyro alignment quaternions are given as

$$\dot{\hat{\mathbf{q}}}_o(t) = \tfrac{1}{2}Q(\hat{\mathbf{q}}_o(t))\tilde{R}_o^T(\tilde{\mathbf{q}}_o(t))(\hat{R}_g(\hat{\mathbf{q}}_g(t))\omega_g(t) + k\tilde{\epsilon}_o(t)\text{sign}(\tilde{\eta}_o(t))) \tag{6a}$$

$$\dot{\hat{\mathbf{q}}}_g(t) = \tfrac{1}{2}Q(\hat{\mathbf{q}}_g(t))\hat{\omega}_g(t) \tag{6b}$$

where $\hat{\mathbf{q}}_o$ represents the transformation from inertial coordinates to the attitude observer coordinates, and $\hat{R}_g(\hat{\mathbf{q}}_g)\omega_g$ is the estimated angular velocity, transformed to observer coordinates with $\tilde{R}_o^T(\tilde{\mathbf{q}}_o)$. The attitude error quaternion, $\tilde{\mathbf{q}}_o$, is computed as in (2) with $\mathbf{q}$ and $\hat{\mathbf{q}}_o$, respectively. In Eq.(6b), $\hat{\mathbf{q}}_g$ represents the transformation from gyro coordinates to an intermediate coordinate frame, denoted as $\hat{\mathbf{b}}$, and $\hat{\omega}_g$ is the angular velocity of the $\hat{\mathbf{b}}$ frame with respect to the gyro frame, which is yet to be specified.

Taking the derivative of $\tilde{\mathbf{q}}_o$, and substituting in Eq. (3) and Eq. (6a) results in the following kinematic equation for the attitude error quaternion

$$\dot{\tilde{\mathbf{q}}}_o(t) = \tfrac{1}{2}Q(\tilde{\mathbf{q}}_o(t))(R_g(\mathbf{q}_g)\omega_g(t) - \hat{R}_g(\hat{\mathbf{q}}_g(t))\omega_g(t) - k\tilde{\epsilon}_o(t)\text{sign}(\tilde{\eta}_o(t))) \tag{7}$$

Define the difference in the two angular velocity terms in Eq. (7) as

$$\tilde{\mathbf{m}}(t) = R_g(\mathbf{q}_g)\omega_g(t) - \hat{R}_g(\hat{\mathbf{q}}_g(t))\omega_g(t) \tag{8}$$

First, consider the case where ω_g is constant. As in Ref. 10, a Lyapunov function is given as

$$V_o(t) = \tfrac{1}{2}\tilde{\epsilon}_o^T(t)\tilde{\epsilon}_o(t) + \begin{cases} \tfrac{1}{2}(\tilde{\eta}_o(t)-1)^2 (\tilde{\eta}_o \geq 0) \\ \tfrac{1}{2}(\tilde{\eta}_o(t)+1)^2 (\tilde{\eta}_o < 0) \end{cases} + \tfrac{1}{2}\tilde{\mathbf{m}}^T(t)\tilde{\mathbf{m}}(t) \tag{9}$$

The derivative of $V_o(t)$ is

$$\dot{V}_o(t) = \tilde{\epsilon}_o^T(t)\dot{\tilde{\epsilon}}_o(t) + \begin{cases} (\tilde{\eta}_o(t)-1)\dot{\tilde{\eta}}_o(t)(\tilde{\eta}_o \geq 0) \\ (\tilde{\eta}_o(t)+1)\dot{\tilde{\eta}}_o(t)(\tilde{\eta}_o < 0) \end{cases} + \tilde{\mathbf{m}}^T(t)\dot{\tilde{\mathbf{m}}}(t) \tag{10}$$

Noting that $\tilde{\epsilon}_o^{\,T}(t)\dot{\tilde{\epsilon}}_o(t) + \tilde{\eta}_o(t)\dot{\tilde{\eta}}_o(t) = 0$ and substituting $\dot{\tilde{\eta}}_o(t)$ from Eq. (7) into Eq. (10) yields

$$\dot{V}_o(t) = -\tfrac{k}{2}\tilde{\epsilon}_o^T(t)\tilde{\epsilon}_o(t) + \tfrac{1}{2}\tilde{\epsilon}_o^T(t)\tilde{\mathbf{m}}(t)\mathrm{sign}(\tilde{\eta}_o(t)) + \tilde{\mathbf{m}}^T(t)\dot{\tilde{\mathbf{m}}}(t) \tag{11}$$

If $\dot{\tilde{\mathbf{m}}}(t)$ is defined as

$$\dot{\tilde{\mathbf{m}}} = -\tfrac{1}{2}\tilde{\epsilon}_o\mathrm{sign}(\tilde{\eta}_o) \tag{12}$$

The derivative of the Lyapunov becomes

$$\dot{V}(t) = -\tfrac{k}{2}\tilde{\epsilon}_o^T(t)\tilde{\epsilon}_o(t) \tag{13}$$

Given that $V_o(t)$ in Eq. (9) is lower bounded, and Eq. (10) shows that $\dot{V}_o(t) \le 0$, $\tilde{\epsilon}_o$ and $\tilde{\mathbf{m}}$ are bounded. $V_o(t)$ is a continuous, twice differentiable function, with $\ddot{V}_o(t)$ bounded. Applying Barbalat's lemma[2] shows that, in fact, $\|\tilde{\epsilon}_o(t)\| \to 0$.

In order to determine a form for $\hat{\omega}_g(t)$ in Eq. (6b), the derivative of $\tilde{\mathbf{m}}$, as given in Eq. (8), is computed as

$$\dot{\tilde{\mathbf{m}}}(t) = -\dot{\hat{R}}_g(\hat{\mathbf{q}}_g(t))\omega_g = S(\hat{\omega}_g(t))\hat{R}_g(\hat{\mathbf{q}}_g(t))\omega_g \tag{14}$$

Equating Eq. (12) and Eq. (14)

$$\tfrac{1}{2}\tilde{\epsilon}_o(t)\mathrm{sign}(\tilde{\eta}_o(t)) = S(\hat{R}_g(\hat{\mathbf{q}}_g(t))\omega_g)\hat{\omega}_g(t) \tag{15}$$

Since $\hat{\omega}_g(t)$ is an observer design variable, it is designed to be perpendicular to $\hat{R}_g(\hat{\mathbf{q}}_g(t))\omega_g$ (any parallel component does not appear in Eq. (15)). This results in

$$\hat{\omega}_g(t) = \tfrac{1}{2}\mathrm{sign}(\tilde{\eta}_o(t))S(\tilde{\epsilon}_o(t))\hat{R}_g(\hat{\mathbf{q}}_g(t))\omega_g$$

The above Lyapunov analysis shows that the errors $\tilde{\epsilon}_o$ and $\tilde{\mathbf{m}}$ are stable and bounded. Therefore, as in Ref. 10, the system can be analyzed as a linear time varying system. Let

$$\begin{bmatrix} \dot{\tilde{\epsilon}}_o(t) \\ \dot{\tilde{\mathbf{m}}}(t) \end{bmatrix} = A(t)\begin{bmatrix} \tilde{\epsilon}_o(t) \\ \tilde{\mathbf{m}}(t) \end{bmatrix}$$

with $A(t) = \begin{bmatrix} -\tfrac{1}{2}k\mathrm{sign}(\tilde{\eta}_o(t))(\tilde{\eta}_o(t)I + S(\tilde{\epsilon}_o(t)) & -\tfrac{1}{2}(\tilde{\eta}_o(t)I + S(\tilde{\epsilon}_o(t)) \\ -\tfrac{1}{2}\mathrm{sign}(\tilde{\eta}_o(t)) & 0 \end{bmatrix}$

As shown in Ref. 10, this system is exponentially stable. In other words, both $\|\tilde{\epsilon}_o(t)\|$ and $\|\tilde{\mathbf{m}}(t)\|$ converge to zero exponentially fast.

The convergence of $\tilde{\mathbf{m}}(t)$ does not guarantee that the gyro alignment estimate $\hat{\mathbf{q}}_g(t)$ converges to the true gyro alignment, $\mathbf{q}_g$. Rewriting Eq. (7) as

$$\tilde{\mathbf{m}}(t) = (I - \tilde{R}_g^T(\tilde{\mathbf{q}}_g(t))\omega$$

Given that $(I - \tilde{R}_g^T)\tilde{\epsilon}_g = 0$, the error, $\tilde{\mathbf{m}}(t)$, is zero when the angular velocity term, $\omega(t)$ is parallel to $\tilde{\epsilon}_g(t)$, or, ideally, when $\|\tilde{\epsilon}_g(t)\| = 0$ indicating that the estimated alignment quaternion is equal to the true alignment quaternion. Given that the convergence of the attitude error is exponential for a constant angular velocity, a series of discrete attitude changes provides observability of the gyro alignment errors, allowing convergence of $\hat{\mathbf{q}}_g$ to $\mathbf{q}_g$.

Next, consider next the case where ω is not constant. In this case, the derivative of $\tilde{\mathbf{m}}(t)$, given in Eq. (8), becomes

$$\dot{\tilde{\mathbf{m}}}(t) = (I - \tilde{R}_g(\tilde{\mathbf{q}}_g(t))\dot{\omega}(t) - \frac{d}{dt}(\tilde{R}_g(\tilde{\mathbf{q}}_g(t))\omega(t)) \qquad (16)$$

Inserting (16) into (11), along with (15) results in

$$\dot{V}_o(t) = -\tfrac{k}{2}\tilde{\epsilon}_o^T(t)\tilde{\epsilon}_o(t) + \tilde{\mathbf{m}}^T(t)\dot{\tilde{\mathbf{m}}}(t) = -\tfrac{k}{2}\tilde{\epsilon}_o^T(t)\tilde{\epsilon}_o(t) + \omega^T(t)(I - \tilde{R}_g(\tilde{\mathbf{q}}_g(t))(I - \tilde{R}_g^T(\tilde{\mathbf{q}}_g(t))\dot{\omega}(t)$$

Using Eq. (1) to expand $\tilde{R}_g(\tilde{\mathbf{q}}_g(t))$ in terms of the quaternion components gives

$$\dot{V}_o(t) = -\tfrac{k}{2}\tilde{\epsilon}_o^T(t)\tilde{\epsilon}_o(t) + \tilde{\epsilon}_g^T(t)(\omega^T(t)\dot{\omega}(t) - \omega(t)\dot{\omega}^T(t))\tilde{\epsilon}_g(t) \qquad (17)$$

If $\omega^T(t)\dot{\omega}(t) \le 0$ due to $\omega(t)$ changing direction and/or decreasing in magnitude, (17) can be rewritten as

$$\dot{V}_o(t) \le -\tfrac{k}{2}\|\tilde{\epsilon}_o(t)\|_2^2 - 8\|\tilde{\mathbf{m}}(t)\|_2^2 \qquad (18)$$

using $\|I - R\|_2 = 2\|\epsilon\|_2$. Since $\dot{V}_o(t) \le 0$, $\|\tilde{\epsilon}_o(t)\|$ and $\|\tilde{\mathbf{m}}(t)\|$ converge to zero exponentially fast. Again, this does not guarantee that $\|\tilde{\epsilon}_g(t)\|$ converges to zero. (The second term on the right of (17) can also evaluate to zero. The convergence then follows from the argument above.)

Integrating (18) results in

$$V_o(t) - V_o(t_o) \le -\frac{k}{2}\int_{t_o}^{t}\left\|\tilde{\epsilon}_o(\tau)\right\|^2 d\tau + \int_{t_o}^{t}\tilde{\epsilon}_g^T(\tau)(\omega^T(\tau)\dot{\omega}(\tau) - \omega(\tau)\dot{\omega}^T(\tau))\tilde{\epsilon}_g(\tau)d\tau \qquad (19)$$

Assuming that $\omega^T(t)\dot{\omega}(t) \le 0$, the integral terms are finite since the integrands converge to zero exponentially fast. If, however, $\omega(t)$ satisfies a 'persistency of excitation' condition, such that

$$-\alpha_2 I \ge \int_{t}^{t+\delta}(\omega^T(t)\dot{\omega}(t) - \omega(t)\dot{\omega}^T(t))d\tau \ge -\alpha_1 I$$

for positive constants α_1, α_2, and δ, the terms in (19) can only remain finite if $\left\|\tilde{\epsilon}_g(t)\right\|$ converges to zero exponentially fast. Intuitively, the alignment is estimated only when $\omega(t)$ changes direction.

SIMULATION RESULTS

The simulation consists of the above observer equations, programmed into Matlab. Several scenarios are tested. The first considers a constant angular velocity, changed discretely at 40 second intervals about the body axes. The initial quaternions are

$$\mathbf{q}_g^T = [1 \;\; 0 \;\; 0 \;\; 0],\; \hat{\mathbf{q}}_g^T = [0 \;\; 0 \;\; 0 \;\; 1],\; \mathbf{q}^T = [0 \;\; 1 \;\; 0 \;\; 0],\; \hat{\mathbf{q}}^T = [0 \;\; 0 \;\; 0 \;\; 1]$$

The initial angular velocity is $\omega_g^T = [1 \;\; 1 \;\; 1]$ rad/sec. Figure 1 shows that the alignment errors converge to zero.

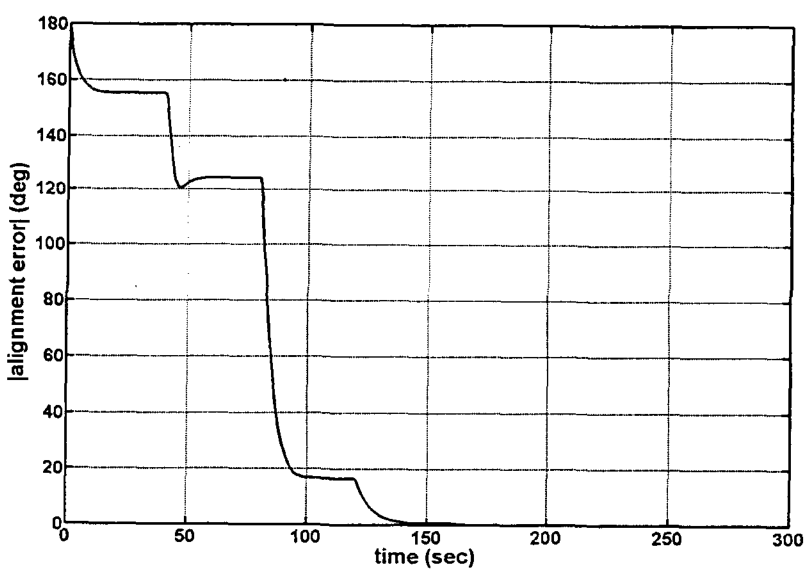

Figure 1. Magnitude of Alignment Errors Given Constant Angular Velocity, With Discrete Changes at 40 Second Intervals

The next tests consider a time varying angular velocity, $\omega(t) = \omega_0 e^{-k_1 t}$. The initial quaternions and ω_0 are as given above. Figure 2 shows that the alignment errors converge to a constant, not to zero.

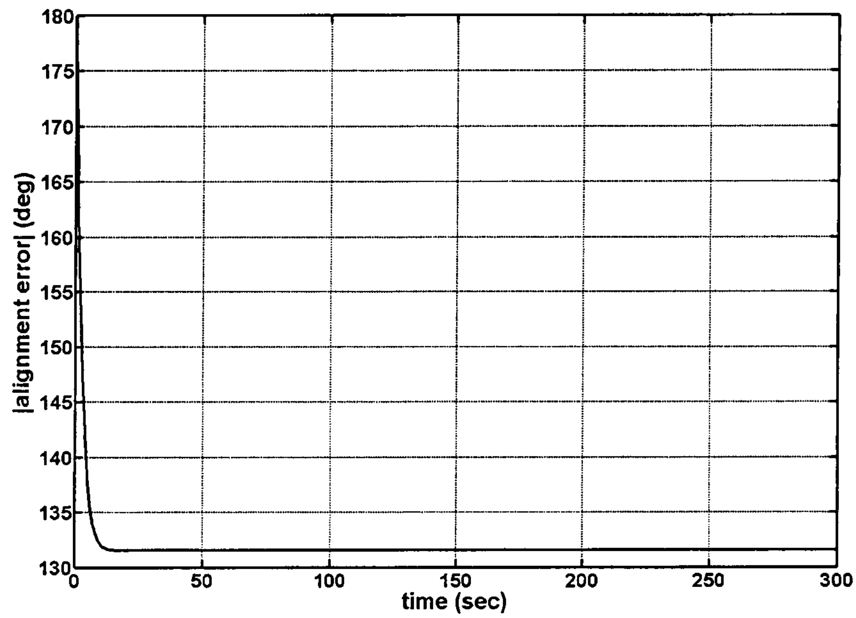

Figure 2. Magnitude of Alignment Errors Given an Exponentially Decaying Magnitude

Next, the angular velocity is given as $\omega(t) = A(t)\omega_0 e^{-k_1 t}$, where A(t) is a rotation matrix, computed from the (arbitrary) 3-1-2 Euler sequence of [0.3t;-0.1t;0.2t]. The initial quaternions and ω_0 are as given above. Figure 3 shows that the alignment errors converge to zero. This test is repeated with random initial quaternions. Figure 4 shows, again, that the alignment errors converge to zero.

Finally, the angular velocity is given as $\omega(t) = A(t)\omega_0$, with A(t) and ω_0 as above. Figures 5 and 6 show the alignment errors given the initial quaternions above, and random initial quaternions, respectively. In both cases, the alignment errors converge to zero.

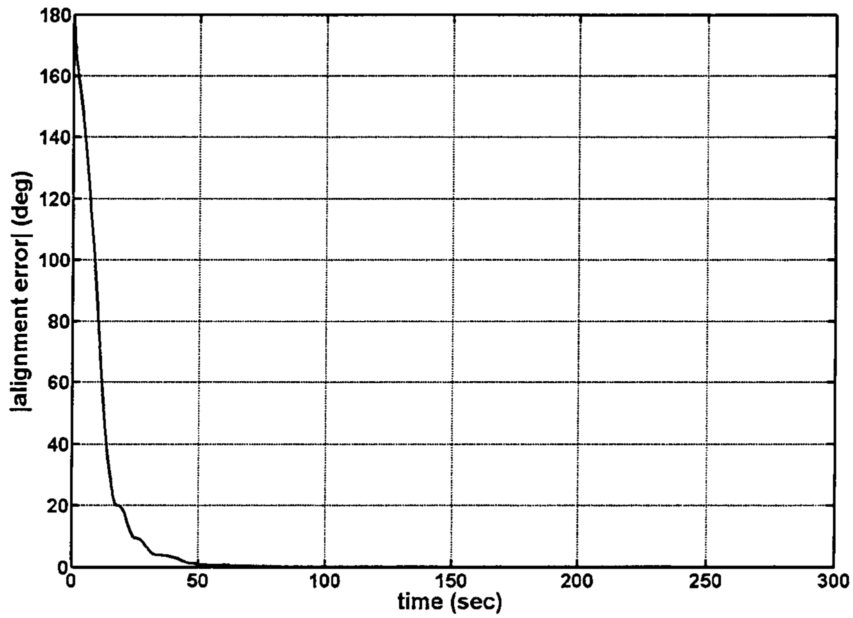

Figure 3. Magnitude of Alignment Errors Given an Exponentially Decaying/Sinusoidal Angular Velocity

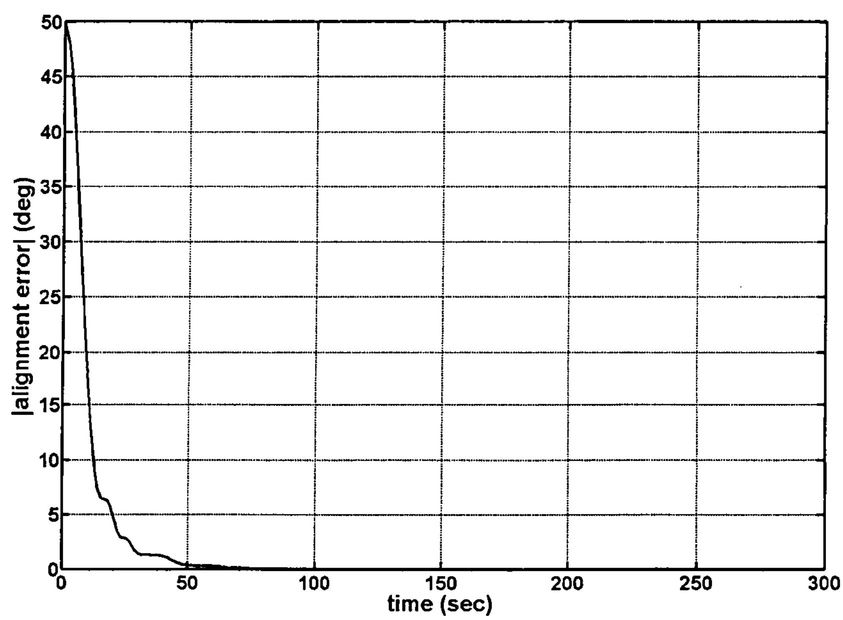

Figure 4. Magnitude of Alignment Errors Given an Exponentially Decaying/Sinusoidal Angular Velocity with Random Initial Quaternions

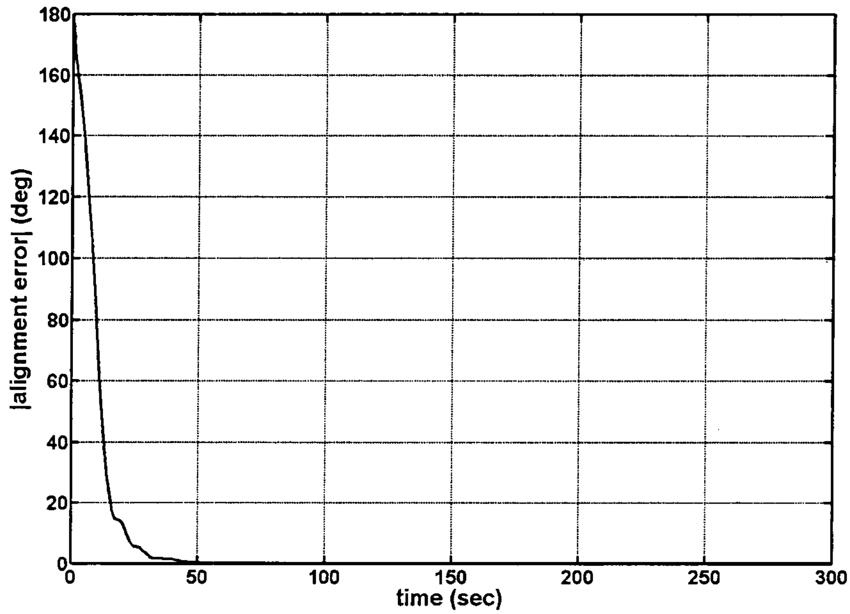

Figure 5. Magnitude of Alignment Errors Given a Sinusoidal Angular Velocity

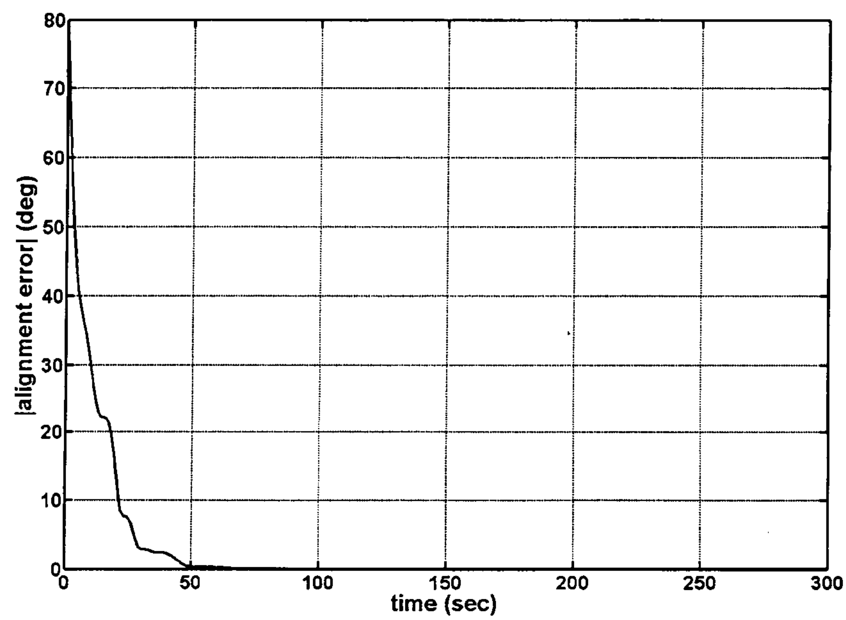

Figure 6. Magnitude of Alignment Errors Given a Sinusoidal Angular Velocity with Random Initial Quaternions

CONCLUSIONS

A nonlinear observer for gyro alignment estimation is developed. The observer is proven, via a Lyapunov argument, to be exponentially stable given a constant angular velocity. The estimated alignment converges to the true alignment when the angular velocity is discretely changed, providing the necessary observability. Given a time varying angular velocity, the estimated alignment converges to the true alignment, when the angular velocity vector changes direction, again to provide the necessary observability.

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